2014-3a

Monday, December 7, 2015 11:14 AM

Let f be analytic on $\{z \in \mathbb{C} : 0 < |z| < 1\}$ satisfying $|f(z)| \leq \log\left(\frac{1}{|z|}\right)$

Suppose it has a Laurent Series

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z^k} + \sum_{l=0}^{\infty} a_l z^l$$

Show that $a_k = 0 \quad \forall \; k = 1, 2, \cdots$

$$a_{-k} = \frac{1}{2\pi i} \int_{C_r} f(\zeta)\, \zeta^{-k+1}\, d\zeta \,, \qquad C_r = \{|\zeta| = r\}, \; 0 < r < 1$$

$$|a_k| \leq \frac{1}{2\pi} \int_{C_r} |f(\zeta)| \cdot |\zeta|^{-k+1} \, |d\zeta|$$

$$\leq \frac{1}{2\pi} \int_{C_r} \frac{\log\left|\frac{1}{\zeta}\right|}{|\zeta|^{k-1}} \, |d\zeta| = \frac{1}{r^{k-1}} \log\frac{1}{r}$$

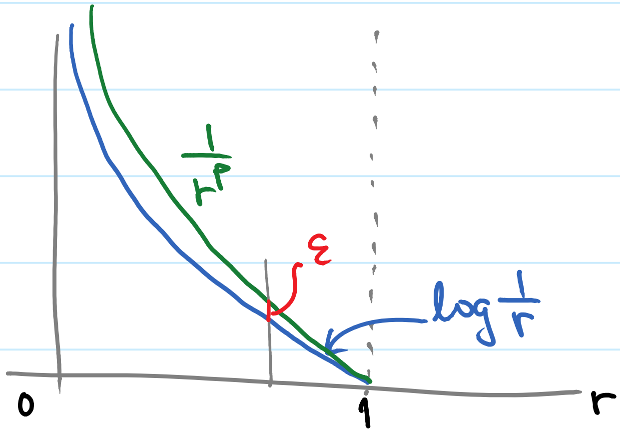
The key is to understand the function

$$r \longmapsto \frac{1}{r^{k-1}} \log\frac{1}{r} \quad \text{for} \quad 0 < r < 1$$

Try to show for each $p = 0, 1, 2, \cdots$

$\forall \; \varepsilon > 0 \;\; \exists \; 0 < r < 1$

$$\frac{1}{r^p} \log\frac{1}{r} < \varepsilon$$



2014-3b,c

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The two parts can be done at the same time.

Now, $f(z) = \sum_{l=0}^{\infty} a_l z^l$, $0 < |z| < 1$

Consider the function

$$\hat{f}(z) = \begin{cases} f(z) & 0 < |z| < 1 \\ a_0 & z = 0 \end{cases}$$

Then $\hat{f}$ is analytic on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

The inequality still holds for $z = 0$

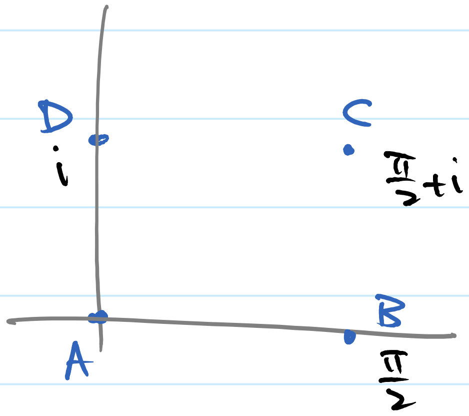
$$|\hat{f}(z)| \le \log\left(\frac{1}{|z|}\right) \qquad 0 \le |z| < 1 .$$

Applying Maximum Principle to $\hat{f}$, try to get that $\hat{f}(z) = 0$ for $|z| = 1$ and indeed $\hat{f}(z) \equiv 0$ for $|z| \le 1$

That implies f is constant zero.

We have $f(z) = \sin z$,

$A = 0$, $B = \frac{\pi}{2}$, $C = \frac{\pi}{2} + i$, $D = i$

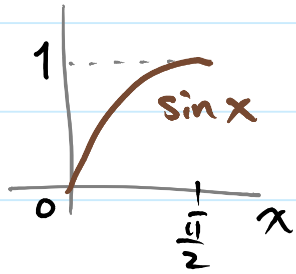


① Image of AB

$AB = \{ x : 0 \leq x \leq \frac{\pi}{2} \}$

$f(AB) = \{ \sin x : 0 \leq x \leq \frac{\pi}{2} \}$

$= [0, 1] \subset \mathbb{C}$ obviously

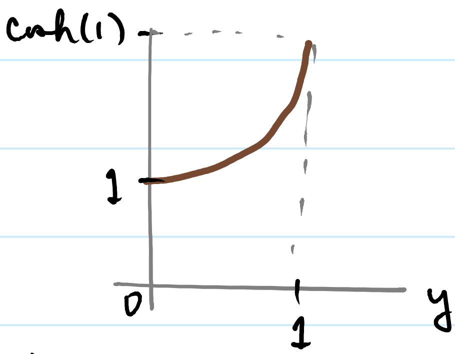


② $BC = \{ \frac{\pi}{2} + yi : 0 \leq y \leq 1 \}$

$f(\frac{\pi}{2} + yi) = \sin\frac{\pi}{2} \cos(yi) + \cos\frac{\pi}{2} \sin(yi)$

$= \cos(yi) = \frac{1}{2}(e^{iyi} + e^{-iyi}) = \cosh(y)$

$f(BC) = [1, \cosh(1)] \subset \mathbb{C}$



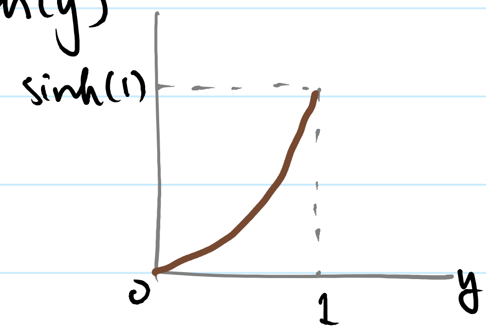
③ $AD = \{ yi : 0 \leq y \leq 1 \}$

$f(yi) = \sin(yi) = \frac{1}{2i}(e^{iyi} - e^{-iyi})$

$= \frac{i}{2}(e^{y} - e^{-y}) = i \sinh(y)$

$f(AD) = i[0, \sinh(1)]$

$\subset$ imaginary axis



2014-4b

Tuesday, December 8, 2015 10:30 AM

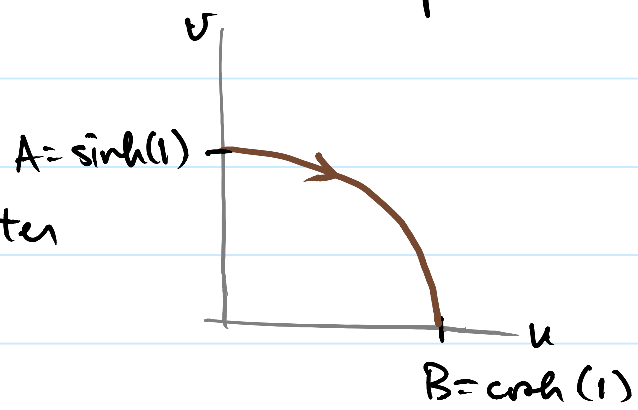
$$CD = \{ x + i \ : \ 0 \leq x \leq \tfrac{\pi}{2} \}$$

$$\begin{aligned} f(x+i) &= \sin(x+i) \\ &= \sin x \cos(i) + \cos x \sin(i) \\ &= \sin x \cosh(1) + i \cos x \sinh(1) \\ &= \quad u(x) + i\, v(x) \end{aligned}$$

$$\left.\begin{aligned} u(x) &= \sin x \cosh(1) = B \sin x \\ v(x) &= \cos x \sinh(1) = A \cos x \end{aligned}\right\} \quad x \in [0, \tfrac{\pi}{2}]$$

This is a parametric equation for an ellipse in the 1^{st} quadrant with u-intercept at $(B, 0)$ and v-intercept $(0, A)$

Thus $f(BC)$ is a quarter of an ellipse



2014-4c

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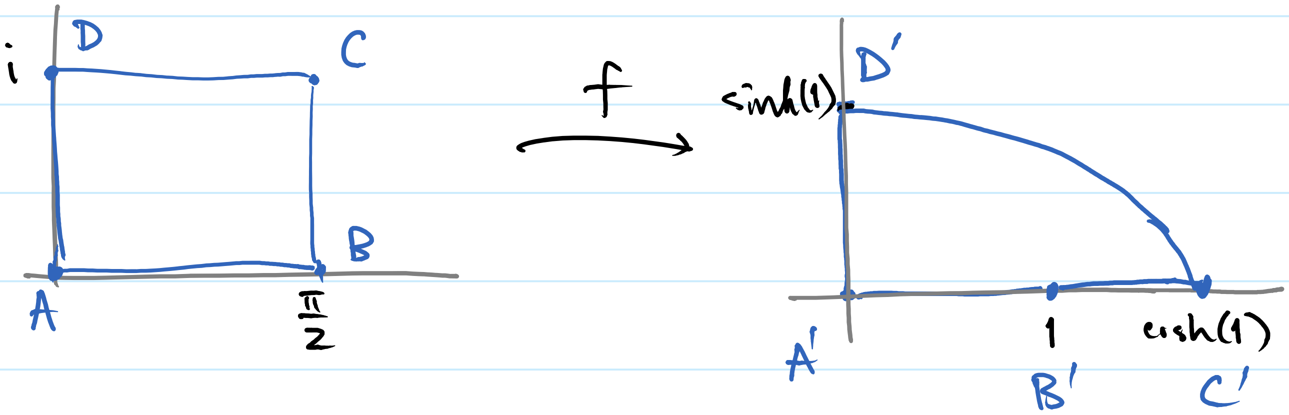
Now, we have the following



Since $f(z) = \sin z$ is conformal and orientation preserving,

$f(ABCD)$ = The region bounded by $u \geq 0$, $v \geq 0$, and

$$\frac{u^2}{\cosh^2(1)} + \frac{v^2}{\sinh^2(1)} \leq 1$$

2014-2

Tuesday, December 8, 2015 10:47 AM

This is a typical question about Argument Principle

It will not be tested in section A.